

# Quark matter in $QC_2D$

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**Abstract.** Results are presented from a numerical study of lattice QCD with gauge group  $SU(2)$  and two flavors of Wilson fermion at non-zero quark chemical potential  $\mu \gg T$ . Studies of the equation of state, the superfluid condensate, and the Polyakov line all suggest that in addition to the low-density phase of Bose-condensed diquark baryons, there is a deconfined phase at higher quark density in which quarks form a degenerate system, whose Fermi surface is only mildly disrupted by Cooper pair condensation.

**PACS.** 11.15.Ha Lattice gauge theory – 21.65.+f Nuclear matter

## 1 Introduction

The phase structure of QCD at large baryon density is one of the most fascinating areas of strong-interaction physics, and yet a systematic calculational approach to this problem remains elusive. Lattice QCD simulation, the usual non-perturbative approach of choice, fails dismally because in Euclidean metric the quark action  $\bar{q}M(\mu)q$ , where  $M = \mathcal{D}[A] + \mu\gamma_0 + m$  with  $\mu$  the quark chemical potential, results in a complex-valued path integral measure  $\det M$  when  $\mu \neq 0$ . Since  $\mu > 0$  promotes baryon current flow in the positive  $t$ -direction, the fundamental reason for this *Sign Problem* can be traced to the explicit breaking of time reversal symmetry. Because the measure no longer has an interpretation as a probability distribution, Monte Carlo importance sampling, the mainstay of lattice simulations, is completely ineffective in the thermodynamic limit.

It is instructive to ask what goes wrong when simulations are performed with a measure  $\det M^\dagger M$  which is positive definite by construction, as is the case for all practical fermion algorithms. It turns out that while  $M$  describes a color-triplet quark  $q \in \mathbf{3}$ ,  $M^\dagger$  describes a *conjugate quark*  $q^c \in \bar{\mathbf{3}}$ . The model's spectrum thus contains gauge-singlet  $qq^c$  states, indistinguishable from mesons at  $\mu = 0$ , but carrying non-zero baryon number. As  $\mu$  rises, baryonic matter first appears in the ground state (*i.e.*  $n_q > 0$ ) at an onset  $\mu_o \sim \frac{1}{2}m_\pi$ , *i.e.* with an energy per quark comparable with the lightest baryon in the spectrum, which is degenerate with the pion, rather than the physically expected  $\mu_o \sim \frac{1}{3}m_{nucleon}$ . Only calculations performed with the correct measure  $\det^{N_f} M$  have cancellations among configurations, due to the fluctuating

phase of the determinant, which ensure that  $n_q$  vanishes for  $\frac{1}{2}m_\pi < \mu < \frac{1}{3}m_{nucleon}$ .

For Two-Color QCD ( $QC_2D$ ), *i.e.* for the gauge group  $SU(2)$ , this bug is actually a feature. Since  $q$  and  $\bar{q}$  live in equivalent representations of the color group, hadron multiplets contain both  $q\bar{q}$  mesons and  $qq$  baryons. It is correspondingly straightforward to show that the quark determinant is positive definite for even  $N_f$  [1].  $QC_2D$  is thus the simplest model of dense strongly interacting matter amenable to study with orthodox lattice techniques. Additionally, if there is a separation of scales  $m_\pi \ll m_\rho$  in the spectrum, then at low densities attention may be focussed on the Goldstone bosons of the system (both mesons and baryons) using chiral perturbation theory ( $\chi$ PT) [2]. The key result is that for  $\mu \geq \mu_o = \frac{1}{2}m_\pi$ , a non-vanishing quark density  $n_q > 0$  develops, along with a superfluid diquark condensate  $\langle qq \rangle \neq 0$ . Just above onset, the system is thus a textbook Bose Einstein Condensate (BEC) formed from tightly bound scalar diquarks.

Using the  $\chi$ PT prediction for  $n_q(\mu)$  [2], it is simple to develop the full equation of state, *i.e.* pressure  $p$  and energy density  $\varepsilon_q$ , at  $T = 0$  [3]:

$$\begin{aligned} n_q &= 8N_f f_\pi^2 \mu \left( 1 - \frac{\mu_o^4}{\mu^4} \right); \\ p &= \int_{\mu_o}^{\mu} n_q d\mu = 4N_f f_\pi^2 \left( \mu^2 + \frac{\mu_o^4}{\mu^2} - 2\mu_o^2 \right); \\ \varepsilon_q &= -p + \mu n_q = 4N_f f_\pi^2 \left( \mu^2 - 3\frac{\mu_o^4}{\mu^2} + 2\mu_o^2 \right); \\ \langle qq \rangle &\propto \sqrt{1 - \frac{\mu_o^4}{\mu^4}}. \end{aligned} \quad (1)$$

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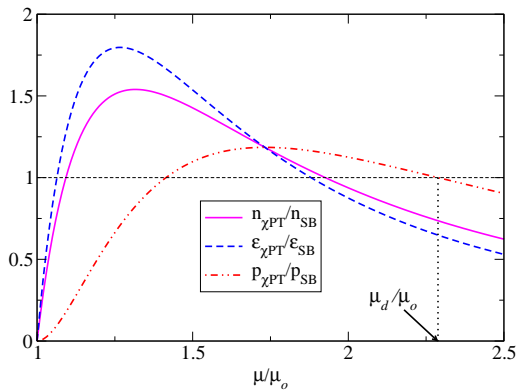


Fig. 1. Model equation of state for  $f_\pi^2 = N_c/6\pi^2$ .

Here,  $f_\pi$  is a parameter of the model. Contrast this with another paradigm for cold dense matter, namely a degenerate system of weakly-interacting massless quarks populating a Fermi sphere up to some maximum momentum  $k_F \approx \mu$ :

$$n_q = \frac{N_f N_c}{3\pi^2} \mu^3; \quad \varepsilon_q = 3p = \frac{N_f N_c}{4\pi^2} \mu^4. \quad (2)$$

Superfluidity in this scenario arises from the condensation of quark Cooper pairs within a layer of thickness  $\Delta$  centred on the Fermi surface, so that  $\langle qq \rangle \propto \Delta \mu^2$ .

Figure 1 plots  $n_q$ ,  $p$  and  $\varepsilon_q$  from (1), each divided by the free field results (2), as functions of  $\mu$ . On equating pressures, this naive model, which ignores all non-Goldstone and gluonic degrees of freedom, predicts a first-order deconfining transition from BEC to “quark matter” at  $\mu_d \approx 2.3\mu_0$  with the choice  $f_\pi^2 = N_c/6\pi^2$ .

## 2 Simulation

To test whether this prediction holds in a more systematic calculation we have performed simulations of  $SU(2)$  lattice gauge theory with  $N_f = 2$  Wilson fermions with  $\mu \neq 0$  [3]. The Wilson formulation is not obviously a stupid choice: Wilson fermions retain a conserved baryon charge; any problems with chiral symmetry should dominate in the low- $k$  region of the quark dispersion curve, which lies at the bottom of the Fermi sea and is hence inert; moreover, studies with free fermions show that saturation artifacts due to the complete filling of the first Brillouin zone actually set in at higher values of  $\mu$  than is the case for staggered fermions [4]. Most importantly, the eigenvalue spectrum of the Wilson-Dirac operator has the same symmetries as that of continuum  $QC_2D$ . As shown in [3], this fact permits an exact ergodic hybrid Monte Carlo algorithm for  $N_f = 2$ , with no requirement to take a fourth root, which may be problematic for  $\mu \neq 0$  [5]. The only novelty of our simulation is the inclusion of a diquark source term

$$jqq \equiv j\kappa(-\bar{\psi}_1(x)C\gamma_5\tau_2\bar{\psi}_2^{tr}(x) + \psi_2^{tr}(x)C\gamma_5\tau_2\psi_1(x)) \quad (3)$$

in the dynamics, where subscripts label flavor and the Pauli matrix acts on color. As well as making the algorithm ergodic, setting  $j \neq 0$  mitigates the effect of IR fluctuations due to Goldstone modes in any superfluid phase, and of course enables direct estimation of the  $\langle qq \rangle$  condensate.

Our initial study has been performed on an  $8^3 \times 16$  lattice using a standard Wilson gauge action, with parameters  $\beta = 1.7$ ,  $\kappa = 0.178$ , and  $j = 0.04$  (with a few points taken at  $j = 0.02, 0.06$ ). Studies of the static quark potential and the hadron spectrum at  $\mu = 0$  yield  $a = 0.220$  fm,  $m_\pi a = 0.79(1)$ , and  $m_\pi/m_\rho = 0.80(1)^1$ . We thus expect the onset of baryonic matter at  $\mu_0 a \approx 0.4$ . Thermodynamic observables are calculated as follows: quark density is given by a local operator via

$$n_q = -\frac{\partial \ln \mathcal{Z}}{\partial \mu}. \quad (4)$$

As a component of a conserved current, it is immune from quantum corrections, but may be affected by artifacts due to  $a > 0$ ,  $V < \infty$ . We therefore prefer to quote our results in terms of  $n_q/n_{SB}^{\text{latt}}$ , where  $n_{SB}^{\text{latt}}(\mu)$  is evaluated for free massless quarks on the same lattice. The pressure follows from an integral formula

$$\frac{p}{p_{SB}} = \int_{\mu_0}^{\mu} \frac{n_{SB}^{\text{cont}}}{n_{SB}^{\text{latt}}} n_q d\mu / \int_{\mu_0}^{\mu} n_{SB}^{\text{cont}} d\mu. \quad (5)$$

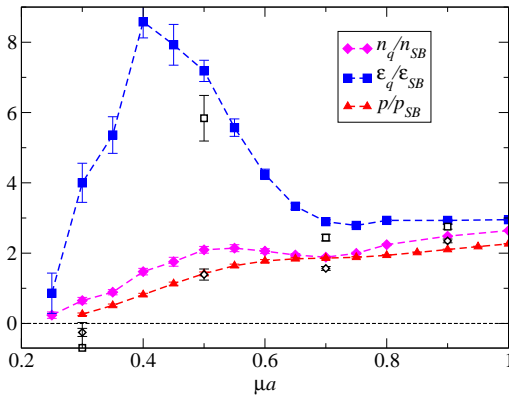
Note that although  $p$  is calculated purely in terms of quark observables, it is in principle the pressure of the system as a whole, although both continuum and thermodynamic limits must eventually be taken. Finally, quark energy density is also estimated by a local operator

$$\varepsilon_q = \kappa \langle \bar{\psi}_x(\gamma_0 - 1)e^\mu U_{0x} \psi_{x+\hat{0}} - \bar{\psi}_x(\gamma_0 + 1)e^{-\mu} U_{0x-\hat{0}}^\dagger \psi_{x-\hat{0}} \rangle; \quad (6)$$

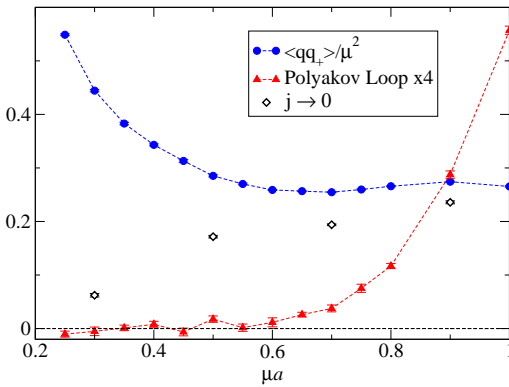
this requires both subtraction of the  $\mu = 0$  vacuum contribution, and a  $\mu$ -independent but as yet unknown multiplicative renormalisation. In what follows, therefore, the shape of the curve is in principle correct, but the overall scale still undetermined.

Figure 2 summarises our results. Both  $n_q$  and  $p$  start to rise from zero at  $\mu a \approx 0.3$ , although a careful  $j \rightarrow 0$  extrapolation will be needed to pinpoint the onset with any precision. By  $\mu a \approx 0.5$  both quantities scale with  $\mu$  in general accordance with free-field predictions, but with approximately twice the expected value. One explanation of this mismatch is that the system has formed a Fermi sphere with  $\mu = E_F < k_F \propto n_q^{\frac{1}{3}}$ , which could be attributed to a negative binding contribution to  $E$  from interactions. The quark energy density, by contrast, increases more slowly than free-field expectations up to  $\mu a \approx 0.65$ , whereupon free-field scaling sets in rather abruptly. Another intriguing result [3] is that for  $0.4 \leq \mu a < 1.0$  the gluon energy density  $\varepsilon_g$  (identically zero in free-field theory) scales to quite high precision as  $\mu^4$ , the only physically sensible possibility once  $\mu/T \gg 1$ . Note that  $\varepsilon_g > 0$

<sup>1</sup> This corrects the value erroneously given in [3].



**Fig. 2.** Lattice equation of state for  $j = 0.04$  (open symbols give  $j \rightarrow 0$  extrapolation).



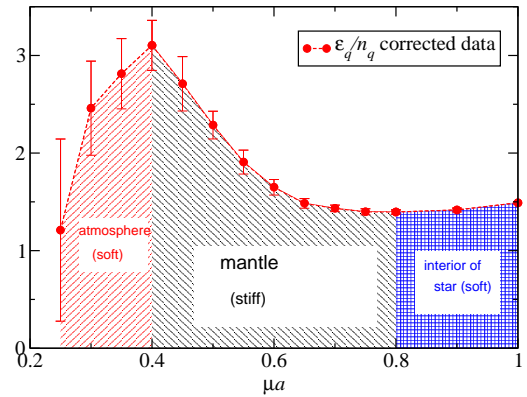
**Fig. 3.** Order parameters  $\langle qq \rangle$  and  $L$  vs.  $\mu$ .

entirely as a result of interactions with the background quark density, since this is the only means by which  $\mu$ -dependence can arise.

To elucidate what is happening, fig. 3 plots both the superfluid order parameter  $\langle qq \rangle$  divided by  $\mu^2$ , and the Polyakov line  $L$ . For  $\mu a \geq 0.5$  it is clear that the system is in a superfluid phase, but what is remarkable is that at  $\mu a \approx 0.6$  there is a sudden transition to a regime where  $\langle qq \rangle \propto \mu^2$ , as expected for BCS pairing at a Fermi surface. At roughly the same point  $L$  rises from zero; although for theories with fundamental matter  $L$  is not strictly an order parameter, this is suggestive that at  $\mu a \approx 0.65$  there is a *deconfining* transition, beyond which the effective degrees of freedom are best thought of as quarks (or even *quasiquarks*) and not the scalar diquarks of  $\chi$ PT.

### 3 Discussion

Our initial study of thermodynamic quantities, and of the properties of the ground state, strongly suggests that QC<sub>2</sub>D at low temperature has at least two transitions as the chemical potential  $\mu$  is raised. The first is between the vacuum and a phase of Bose-condensed tightly bound diquarks; the second, a relativistic analogue of the BEC/BCS crossover currently discussed in both strongly



**Fig. 4.** Energy per quark  $\epsilon_q/n_q$  vs.  $\mu$ .

correlated electron and cold atom systems, is a deconfining transition to a system of degenerate quarks, the Fermi surface being mildly disrupted by a Cooper pair condensate. Although QC<sub>2</sub>D clearly models nuclear matter unrealistically, its description of quark matter may well prove to have much in common with that of QCD. We are currently extending our study to the hadron spectrum, and to finer lattice spacings to check that this conclusion is not due to lattice artifacts. Interesting results obtained from a study of the gluon propagator on the current system will be discussed elsewhere [6, 3].

Meanwhile it is hard to resist the temptation to speculate on what a two-color star might look like. Figure 4 plots the energy per quark  $\epsilon_q/n_q$  vs.  $\mu$  using the data of fig. 2. The most striking feature of this plot is the pronounced minimum at  $\mu a \approx 0.8$ , which is both robust (since it occurs even if corrections for  $a > 0$ ,  $V < \infty$  are left out), and unexpected (since it does not occur for the model EoS of fig. 1). We infer that any large object assembled from a fixed number of QC<sub>2</sub>D quarks, such as a star, will have the bulk of its interior in the neighbourhood of this minimum, which as fig. 3 shows, means that the object would in effect be a quark star formed from deconfined matter. Somewhat speculatively, we have labelled the different regions of the  $\mu$ -axis with the corresponding layers of the star, although a quantitative solution for the radial profile must await correctly normalised calculations of the energy densities  $\epsilon_q$  and  $\epsilon_g$ .

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